THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 4
January 15, 2025 (Wednesday)

1 Recall

In the previous lectures, we introduce the *feasible set* K as follows:

$$\boxed{K = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ h_j(x) = 0,\ i = 1, \ldots, \ell,\ j = 1, \ldots, m\}}$$

Yesterday, we had discussed the Qualification condition as well:

> We say the constraints K is **qualified** at $x \in K$ if $p_i \geq 0$ and $q_j \in \mathbb{R}$ satisfy
>
> $$\begin{cases} \sum_{i=1}^{\ell} p_i g_i(x) = 0 \\ \sum_{i=1}^{\ell} p_i \nabla g_i(x) + \sum_{j=1}^{m} q_j \nabla h_j(x) = \mathbf{0} \end{cases}$$
>
> then it implies that $p_1 = \cdots = p_\ell = q_1 = \cdots = q_m = 0$.

and the Theorem related to optimal solution x^* and qualified K is shown as follows:

> Let $x^* \in K$ be a solution to (P) and assume that K is **qualified** at x^*. Then there exists $\lambda_1, \cdots, \lambda_\ell \geq 0$ and $\mu_1, \cdots, \mu_m \in \mathbb{R}$ such that
>
> $$\begin{cases} \sum_{i=1}^{\ell} \lambda_i g_i(x^*) = 0 \\ \nabla f(x^*) + \sum_{i=1}^{\ell} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{m} \mu_j \nabla h_j(x^*) = \mathbf{0} \end{cases}$$

2 Checking Qualification Condition

Let us introduce a proposition first.

Proposition 1. *Let $x \in K$ satisfies the followings (**Mangasarian Fromovitz condition**)*

1. *the family of vectors $(\nabla h_1(x), \ldots, \nabla h_m(x))$ is linearly independent.*
2. *there exists a vector $v \in \mathbb{R}^n$ satisfying*

   $$\langle \nabla h_j(x), v \rangle = 0,\ \forall j = 1, \ldots, m$$

   and

   $$\langle \nabla g_i(x), v \rangle < 0,\ \forall i \in I(x) := \{k : g_k(x) = 0\}.$$

   Then the constraint K is qualified at $x \in K$.

Remarks. Note that

$$\begin{aligned} & (\nabla h_1(x), \dots, \nabla h_m(x)) \quad \text{is linearly independent} \\ \iff & \sum_{j=1}^{m} q_j \nabla h_j(x) = 0 \implies q_j = 0, \forall j = 1, \dots, m \\ \iff & \operatorname{Rank}(\nabla h(x)) = m \end{aligned}$$

Proof. Let $p_1, \dots, p_\ell \geq 0,\ q_1, \dots, q_m \in \mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{\ell} p_i g_i(x) = 0 \iff p_i g_i(x) = 0,\ \forall i = 1, \dots, \ell \\ \displaystyle\sum_{i=1}^{\ell} p_i \nabla g_i(x) + \sum_{j=1}^{m} q_j \nabla h_j(x) = \mathbf{0} \end{cases}$$

Define $I(x) := \{i = 1, \dots, \ell : g_i(x) = 0\}$ be the index set. Then, the equaliy holds would imply that

$$p_i = 0,\ \forall i \notin I(x) \quad \text{because} \quad g_i(x) < 0$$

Hence, we have

$$\begin{aligned} & \implies \sum_{i \in I(x)} p_i \nabla g_i(x) + \sum_{j=1}^{m} q_j \nabla h_j(x) = \mathbf{0} \\ & \implies \sum_{i \in I(x)} \underbrace{p_i}_{\geq 0} \underbrace{\langle \nabla g_i(x), v \rangle}_{<0} + \sum_{j=1}^{m} q_j \underbrace{\langle \nabla h_j(x), v \rangle}_{=0} = 0 \end{aligned}$$

So, it follows that

1. $\forall i \notin I(x)$, we have $p_i = 0$.
2. $\forall i \in I(x)$, we also have $p_i = 0$.

and thus

$$\sum_{i=1}^{\ell} \underbrace{p_i}_{=0} \nabla g_i(x) + \sum_{j=1}^{m} q_j \nabla h_j(x) = \sum_{j=1}^{m} q_j \nabla h_j(x) = 0$$

and $q_1 = q_2 = \cdots = q_m = 0$ (by linearly independence). $\square$

Let us go through an example and check the qualification condition by using the above proposition.

Example 1. Discuss the qualification condition for the problem $\displaystyle\min_{x^2+y^2=1} 2x + y$.

Solution. From previous, we know $\ell = 0,\ m = 1$ and letting $h(x, y) = x^2 + y^2 - 1$.
This follows that $\nabla h(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$. It is clear that $\nabla h(x, y) \neq \mathbf{0}$ as $x^2 + y^2 = 1$.
So $\left\{ \begin{pmatrix} 2x \\ 2y \end{pmatrix} \right\}$ is linear independent, the first item of the proposition is satisfied.
Moreover, letting $v = \begin{pmatrix} -y \\ x \end{pmatrix}$, then we verify that

$$\left\langle \begin{pmatrix} 2x \\ 2y \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix} \right\rangle = 0$$

for all $(x, y) \in \mathbb{R}^2$ which also satisfied the second item of the proposition.
By the above proposition, K is qualified at each point $(x, y) \in K$. ◀

Example 2. Discuss the qualification condition for the problem $\min\limits_{x^2+y^2\le 1} xy$.

Solution. Clearly, we have $\ell = 1, m = 0$ and let $g(x, y) = x^2 + y^2 - 1$.
Then, we have $\nabla g(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.
Letting $v = -\begin{pmatrix} 2x \\ 2y \end{pmatrix} \in \mathbb{R}^2$, and we compute

$$\left\langle \nabla g(x, y), -\begin{pmatrix} 2x \\ 2y \end{pmatrix} \right\rangle = -(4x^2 + 4y^2) < 0$$

if $(x, y) \neq (0, 0)$.
By the above proposition, we conclude that K is qualified at each points except **0**. ◀

Example 3. Discuss the qualification for the problem

$$\min_{\substack{x^2+y^2=1 \\ y^2+z^2=4}} x + z$$

Solution. Since $\ell = 0,\ m = 2$, letting $h_1(x, y, z) = x^2 + y^2 - 1$ and $h_2(x, y, z) = y^2 + z^2 - 4$.
Then, we have

$$\nabla h_1 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}, \quad \nabla h_2 = \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix}$$

Note that if $z = 0$, then $y = \pm 2$ and both contradicting to $x^2 + y^2 = 1$.
For $z \neq 0$, then we have $\mathrm{Rank} \begin{pmatrix} 2x & 0 \\ 2y & 2y \\ 0 & 2z \end{pmatrix} = 2$. To check the first item, it is easy that $\{\nabla h_1, \nabla h_2\}$ is linearly independent.
For the second item, as the underlying space is $\mathbb{R}^3$, there exists $v \in \mathbb{R}^3$ such that

$$v \perp \mathrm{Span}(\nabla h_1, \nabla h_2) \iff \langle v, \nabla h_i \rangle = 0,\ \forall i = 1, 2$$

By the previous proposition, K is qualified at all points $(x, y, z) \in K$. ◀

— End of Lecture 4 —